PERTURBED MOTION OF A SYMMETRIC VORTEX STREET

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Consider two parallel rows of point vortices, positioned so that vortices in different rows are aligned. The intensity of the vortices in the rows is Γ and −Γ, and the distances ℓ between the vortices in each row are the same. It is known that such a configuration (symmetric vortex street) is unstable relative to small displacements of the vortices [1]. In this work we shall study the motion of a symmetric vortex street, caused by a perturbation such that every odd vortex pair (the two closest oppositely oriented vortices) converges or diverges by the same amount and is simultaneously displaced along or opposite to the direction of motion of the street also by the same amount, and in addition the convergence or divergence of the vortex pairs does not destroy the symmetry of the vortices relative to an axis passing through the center of the undisturbed vortex pairs. By virtue of the symmetry of the starting configuration and the spatial periodicity of the perturbations, two other symmetric vortex streets, one of which is formed by the initially disturbed vortex pairs and the other by the undisturbed pairs, is separated from the starting street. All further motion will be determined by the nature of the interaction of these streets.

Let us examine one such street. The distance between the vortices in the rows equals 2ℓ, and the distance between the rows equals 2h. The coordinate system is shown in Fig. 1. The complex potential of such a street [1] is given by

$$\omega(z)=\frac{\Gamma}{2\pi i}\left[\ln\sin\frac{\pi(z-z_0)}{2l}-\ln\sin\frac{\pi(z-\xi_0)}{2l}\right],$$
$$z=x+iy,\ z_0=ih,\ \xi_0=-ih.$$

The complex velocity at the point z is given by

$$v_x-iv_y=d\omega/dz,$$

whence we find

$$v_x(x,y)=\frac{\Gamma}{4l}\left[\frac{\operatorname{sh}\frac{\pi(y+h)}{l}}{\operatorname{ch}\frac{\pi(y+h)}{l}-\cos\frac{\pi x}{l}}-\frac{\operatorname{sh}\frac{\pi(y-h)}{l}}{\operatorname{ch}\frac{\pi(y-h)}{l}-\cos\frac{\pi x}{l}}\right];\tag{1}$$

$$v_y(x,y)=\frac{\Gamma}{4l}\sin\frac{\pi x}{l}\left[\frac{1}{\operatorname{ch}\frac{\pi(y-h)}{l}-\cos\frac{\pi x}{l}}-\frac{1}{\operatorname{ch}\frac{\pi(y+h)}{l}-\cos\frac{\pi x}{l}}\right].\tag{2}$$

In the general case the streets will be arranged as shown in Fig. 2 (since the configuration is symmetric relative to the x axis, only vortices lying i the top half-planes are shown). As follows from (1) and (2), one street, left to itself, will move without change of form along the x axis with the velocity [1]

$$V=v_x(0,h)=\frac{\Gamma}{4l}\operatorname{cth}\frac{\pi h}{l}.\tag{3}$$

We denote the velocity created by the i-th street (i = 1, 2) at the point (x, y), by v(x, y; i). Then

$$\dot{x}_1=V(h_1)+v_x(x_1,\ h_1;\ 2);\tag{4}$$

$$\dot{x}_2=V(h_2)+v_x(x_2,\ h_2;\ 1);\tag{5}$$

$$\dot{h}_1=v_y(x_1,\ h_1;\ 2);\tag{6}$$

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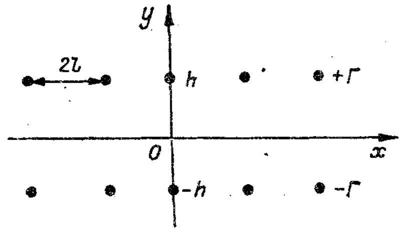

Fig. 1

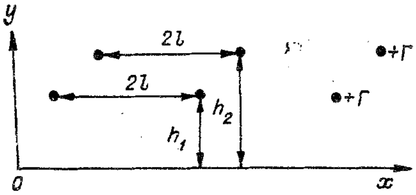

Fig. 2

$$\dot{h}_2 = v_y(x_2, h_2; 1). \tag{7}$$

An overdot indicates differentiation with respect to time. Substituting (2), Eqs. (6) and (7) yield the integral of motion

$$h_1 + h_2 = A = \text{const}. \tag{8}$$

Setting $H = x_2 - x_1$, $\delta = h_1 - A/2$, subtracting (4) from (5), and using (1)-(3), (6), and (8), we obtain

$$\dot{H} = \frac{\Gamma}{2l}\,\mathrm{sh}\,\frac{2\pi\delta}{l}\left[\frac{1}{a - \mathrm{ch}\,\frac{2\pi\delta}{l}} + \frac{1}{\mathrm{ch}\,\frac{2\pi\delta}{l} - \cos\frac{\pi H}{l}}\right]; \tag{9}$$

$$\dot{\delta} = \frac{\Gamma}{4l}\sin\frac{\pi H}{l}\left[\frac{1}{a - \cos\frac{\pi H}{l}} - \frac{1}{\mathrm{ch}\,\frac{2\pi\delta}{l} - \cos\frac{\pi H}{l}}\right],\quad a = \mathrm{ch}\,\frac{\pi A}{l}. \tag{10}$$

Denoting $Y = \cos\frac{\pi H}{l}$, $X = \mathrm{ch}\,\frac{2\pi\delta}{l}$ and dividing (9) by (10) we have

$$\frac{dY}{dX} = \left(\frac{Y - a}{X - a}\right)^2. \tag{11}$$

From here we find one other integral of motion

$$\frac{1}{\cos\frac{\pi H}{l} - a} - \frac{1}{\mathrm{ch}\,\frac{2\pi\delta}{l} - a} = C. \tag{12}$$

The point of the (H, δ) plane representing the state of the system under study moves along the corresponding phase trajectory, determined by (12). It is easy to see that these phase trajectories are symmetric relative to the H axis and any straight line $H = n\ell (n = 0, \pm 1, \pm 2, \ldots)$. Since $-A/2 < \delta < A/2$, in order to construct the phase-plane diagram it is sufficient to study the behavior of the phase trajectories in the region $P = \{0 < H < \ell, 0 < \delta < A/2\}$. Dividing (10) by (9), we obtain

$$\frac{d\delta}{dH} = -\frac{\sin\frac{\pi H}{l}}{2\,\mathrm{sh}\,\frac{2\pi\delta}{l}}\left(\frac{\mathrm{ch}\,\frac{2\pi\delta}{l} - a}{\cos\frac{\pi H}{l} - a}\right)^2. \tag{13}$$

It is evident from (13) that everywhere in the region P, $d\delta/dH < 0$. From here it follows that a priori there are two possibilities for the phase trajectory passing through the point $(0, \delta)$: either it intersects the H axis on the segment $(0, \ell)$ and is closed as a result of the symmetry relative to the $H\delta$ axes, or it does not intersect the H axis on this segment and as a result relative of the symmetry to the straight lines $H = n\ell\,(n = 0, \pm 1, \pm 2, \ldots)$ it does not intersect the H axis anywhere, i.e., it is not closed. Since in the region P there are no singular points of (13), the phase trajectories in this region do not intersect one another [2]. For this reason the phase trajectory passing through the singular point $(\ell, 0)$ divides P into two regions, one of which is filled with the closed phase trajectories and the other is filled with open trajectories. Such a phase trajectory is called a separatrix [2]. From (12) we find the point δ_1 at which the separatrix intersects the δ axis:

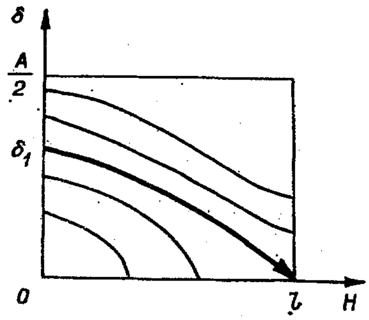

Fig. 3

$$\delta_1 = \frac{l}{2\pi}\,\mathrm{arch}\,\frac{3a+1}{a+3}.$$

It is not difficult to show that $0 < \delta_1 < A/2$. The general form of the phase plane diagram for $0 < H < \ell$, $0 < \delta < A/2$ is presented in Fig. 3. The dark solid line is the separatrix.

If the point representing the state of the system under study moves along a closed phase trajectory, then the quantities H and δ are periodic functions of time [2]. In this case the vortex pairs in the street combine into coupled systems: the back pair converges and is accelerated, and the front pair diverges and is decelerated until the back pair passes through the front pair, after which the pair which is now in front begins to diverge and slow down, and the trailing pair begins to converge and accelerate, after which the process just described repeats, etc. The nature of the motion corresponding to an open phase trajectory is different: the front pair diverges and slows down, and the trailing pair converges and is accelerated, and after passing through the front pair the trailing pair interacts in the same manner with the next pair, etc.

Thus we can say that the symmetric vortex street, being unstable, in the presence of certain perturbations can pass through different types of ordered motions. So, in the first case the entire vortex street separates into cells, consisting of two bound vortex pairs, and in the second case one street moves within the other.

We shall now establish the relationship between the initial perturbation and the nature of the subsequent motion of the system under study. The separatrix passes through the point $(\ell, 0)$ and starting from this and (12) we write its equation in the form

$$\frac{1}{\cos\frac{\pi H}{l} - a} - \frac{1}{\mathrm{ch}\frac{2\pi\delta}{l} - a} = C_1, \quad C_1 = 2/(a^2 - 1). \tag{14}$$

Any phase trajectory lying below the separatrix corresponds to periodic motion, and any trajectory lying above it corresponds to nonperiodic motion (see Fig. 3). It is not difficult to show that for $C < C_1$ the street will separate into cells consisting of bound vortex pairs, and for $C > C_1$ there arises a motion of one street within the other. Here C is the constant appearing in (12) and depends on the initial conditions.

Assume that initially the odd vortex pairs have moved relative to the even pairs along the x axis by an amount ξ and simultaneously every vortex in the odd pairs has moved along the y axis by an amount ε (vortices of different intensities move along the y axis in opposite directions, so that the symmetry of their arrangement relative to the x axis is preserved). From (8) and (12) we find

$$C = \frac{1}{a - \mathrm{ch}\frac{\pi\varepsilon}{l}} - \frac{1}{\cos\frac{\pi\xi}{l} + a}, \quad A = 2h - \varepsilon. \tag{15}$$

Positive values of ε correspond to convergence of the disturbed pairs, and negative values correspond to divergence. Now, in order to determine the nature of the motion occurring after the action of the perturbation described, it is sufficient to compare (14) and (15).

Let, for example, the vortices of the odd pairs move initially along the x axis. It is easy to show that in this case $C < C_1$ for any value of ξ. If the disturbance is such that vortices of odd pairs move along the y axis only, then $C > C_1$ irrespective of the value of ε, i.e., purely longitudinal disturbances give rise to the formation of cells in the street, while purely transverse disturbances separate the starting street into two streets.

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2. A. A. Andropov, A. A. Vitt, and S. E. Khaikin, Theory of Oscillations [in Russian], Fizmatgiz, Moscow (1959).

A METHOD FOR THE SOLUTION OF NONSTATIONARY PROBLEMS FOR A LAYER OF LIQUID WITH MIXED BOUNDARY CONDITIONS

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In this paper we develop a method for the solution of nonstationary problems with mixed boundary conditions for a layer of heavy liquid. In contrast to well-known analytical-numerical approaches (see [1, 2]), the method we propose makes it possible, being based on a factorization method, to carry out an analytical study of the process of excitation and the establishment of waves.

By way of illustrating, we consider the problem of generating excitations by means of a set of external pressures applied to the upper boundary of a layer of liquid partially covered by an eleastic plate. We model a nonstationary process involving the interaction of waves, excited through baric formations, with a limited ice field.

Mathematically stated, our problem has the form

$$\partial \mathbf{v}/\partial t = -\rho_*^{-1}\nabla p_*, \quad -\infty < x, y < \infty, \ -H \leqslant z \leqslant 0, \ \operatorname{div}\mathbf{v} = 0; \tag{1}$$

$$z = 0, \ p_* = q + \rho_* g\zeta + \begin{cases}\Pi\zeta, & \mathbf{x}\in\Omega,\\ 0, & \mathbf{x}\notin\Omega,\end{cases} \quad w = \partial\zeta/\partial t; \tag{2}$$

$$R = M_n = M_\tau = 0, \ \mathbf{x}\in\partial\Omega, \ \mathbf{x} = \{x, y\}; \tag{3}$$

$$\Pi = d_0\nabla^4 + \rho_0 h\frac{\partial^2}{\partial t^2}, \ q = \begin{cases}q(\mathbf{x}, t), & \mathbf{x}\in D, \ t > 0,\\ 0, & \mathbf{x}\notin D,\end{cases}$$

$$z = -H, \quad w = 0;$$

$$t = 0, \left\{\mathbf{v}, q \ \zeta, \ \frac{\partial\zeta}{\partial t}\right\} = 0. \tag{4}$$

Here {x, y, z} is a rectangular Cartesian coordinate system with origin on the unperturbed free surface of the liquid; the z axis is directed vertically upwards; t is the time; p_* is the dynamic component of the total pressure p in the liquid; $v = \{u, v, w\}$ is the velocity vector; ρ_* and H are the density and thickness of the layer of liquid; ζ is the elevation of the free surface, coinciding in the domain Ω occupied by the plate with its vertical displacement. R, M_n, and M_τ are the intersecting force, the bending moment, and the torque on the end of the plate $\partial\Omega$; d_0, ρ_0, and h are the stiffness, the density, and the thickness of the plate; q(x, t) is the external perturbing pressure, specified in the domain $D = D_1 \cup D_2$, acting on the free surface of the liquid in the domain D_1 and on the plate in D_2; g is the gravitational acceleration.

We introduce dimensionless variables, identifying them with the subscript 1:

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